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Isospin-symmetry-breaking corrections to superallowed Fermi β decay: Radial excitations

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Based on an exact formalism, we study the effects of isospin-symmetry-breaking interactions on superallowed $0^+ \to 0^+$ transitions. We calculate the second-order renormalization of the Fermi matrix element due to radial contributions and show that radial excitations neglected in the treatment of Towner and Hardy are significant. These are estimated to decrease the isospin-symmetry-breaking corrections. Our results provide a correction term that can be included in existing approaches.

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I. INTRODUCTION

Superallowed nuclear β decays play a key role for precision tests of fundamental symmetries: They provide the most stringent test of the conserved-vector-current (CVC) hypothesis, the best limit on scalar interactions, and the most precise value for the up-down Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{ud} [1]. The precise extraction of V_{ud} from superallowed transitions requires the evaluation of theoretical corrections due to isospin-symmetry breaking (ISB) [2, 3] and radiative [4] effects with an uncertainty of 10%, to guarantee a desired accuracy of 0.1% for V_{ud}. Because of the very high precision reached experimentally, the uncertainty of V_{ud} is currently limited by ISB and radiative corrections [1]. This presents a challenge for nuclear theory.

Superallowed $0^+ \to 0^+$ Fermi β decays depend only on the vector part of weak interactions, and with CVC, the transition "ft value" should be nucleus-independent:

$$ft = \frac{2\pi^3 \hbar^7 \ln 2}{|M_F|^2 \, G_V^2 m_e^5 c^4} = \text{const.}\,, \tag{1}$$

where G_V is the vector coupling constant and M_F is the Fermi matrix element. The CVC hypothesis depends on the assumption of isospin symmetry, which is not exact in nuclei, but broken by electromagnetic and quark mass effects. As a result, M_F is reduced from the symmetry value, $M_0 = \sqrt{2}$ for $T = 1$ parent and daughter states, by ISB corrections δ_C,

$$|M_F|^2 = |M_0|^2 \, (1 - \delta_C)\,. \tag{2}$$

In addition, there are radiative corrections, but we focus on δ_C in this paper.

Towner and Hardy (TH) have shown [1, 2] that the calculated ISB corrections eliminate much of the considerable scatter present in the uncorrected ft values, and the statistical consistency among the corrected $\mathcal{F}t$ values is some evidence that the calculated ISB corrections are not unreasonable. However, the precise extraction of V_{ud} and the importance of testing the Standard Model have stimulated us to undertake a reevaluation based on an exact formalism [5]. In addition, our goal is to connect the calculated corrections to the accurate understanding of ISB in nuclear forces [6, 7]. Reference [5] showed that there are specific corrections to the treatment of TH [2] and explained these using schematic models without making numerical estimates. In particular, radial excitations are neglected by TH, and as a result the transition operator violates the isospin commuation relations. The TH approach is discussed in more detail in Sect. II.

Following our work, there have been a number of theoretical developments: Auerbach studied ISB corrections assuming that the dominant physics is due to the isovector monopole resonance [8]. Assuming that certain reduced matrix elements of the isosvector part of the Coulomb potential are identical, Auerbach showed that ISB corrections vanish unless one takes into account the energy differences between components of the isovector monopole state of different isospin. Using schematic models, the resulting estimates for δ_C are considerably smaller than the TH results. Liang *et al.* carried out random-phase-approximation calculations based on relativistic density functionals. They also obtain smaller values for δ_C and their corrected $\mathcal{F}t$ values are statistically consistent [9]. In addition, Satula *et al.* have analyzed isospin mixing and implemented isospin projection in density-functional calculations [10].

At the same time, the experimental precision has been improved in several cases [1] and a recent branching ratio measurement for ^{32}Ar [11] has improved the ft value for this $T = 2$, $T_z = -2$ decay to 0.8%, a precision that is nearing the 0.3% standard for inclusion into the set of well-known $T = 1$ decays. This is especially interesting, because ISB corrections appear to be larger for $T = 2$ superallowed transitions.

In this work, we start from the exact formalism developed in Ref. [5], which is reviewed in Sect. III. In Sect. IV we write the correct isospin operator as a sum of the TH operator and a correction term due to radial excitations. We show that the neglected radial contributions are sig-

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nificant at leading (second) order in ISB interactions. In addition, for certain conditions the correction term cancels ISB corrections to Fermi matrix elements. This is similar to the case of Auerbach when the components of the isovector monopole state are degenerate [8]. Our results demonstrate explicitly the importance of radial excitations and we discuss the implications in Sect. V.

II. TH APPROACH TO ISB CORRECTIONS

With CVC, the matrix elements of weak vector interactions in nuclei are not modified by nuclear forces, except for corrections due to ISB effects. Therefore, one has to evaluate the contributions from electromagnetic and charge-dependent strong interactions to the Fermi matrix element $M_F = \langle f|\tau_+|i\rangle$ between the initial and final states for superallowed β decay, $|i\rangle$ and $|f\rangle$, respectively. Here τ_+ denotes the isospin raising operator.

Towner and Hardy [2] use a second-quantized formulation to write the Fermi matrix element as

$$M_F = \sum_{\alpha,\beta} \langle f|a^\dagger_\alpha a_\beta|i\rangle\langle\alpha|\tau_+|\beta\rangle\,, \tag{3}$$

where $a^\dagger_\alpha$ creates a neutron in state α and a_β annihilates a proton in state β. Thus, the label α is used to denote neutron creation and annihilation operators, while β is used for those of the proton. This notation is different from the standard notation [12], in which b_α is used to denote proton annihilation operators. The single-particle matrix element $\langle\alpha|\tau_+|\beta\rangle$ is assumed to be given by

$$\langle\alpha|\tau_+|\beta\rangle = \delta_{\alpha,\beta}\int_0^\infty R^n_\alpha(r)\,R^p_\beta(r)\,r^2 dr \equiv \delta_{\alpha,\beta}\,r_\alpha\,, \tag{4}$$

where $R^n_\alpha(r)$ and $R^p_\beta(r)$ are the neutron and proton radial wave functions, respectively. Because the radial quantum numbers of the states α and β are set to be the same, Eq. (4) assumes that τ_+ creates a neutron in the single-particle state with the same quantum numbers as those of the annihilated proton. This is not the case in the presence of ISB. As a result, the operator in Eq. (3) is not the correct isospin operator and the Standard Model isospin commutation relations are lost.

We observe that Eqs. (3) and (4) correspond to the second-quantized isospin operators

$$\tau_+ \;=\; \sum_{\alpha,\beta}\delta_{\alpha,\beta}\,r_\alpha\,a^\dagger_\alpha a_\beta\,, \tag{5}$$

$$\tau_- \;=\; \tau_+^\dagger = \sum_{\alpha,\beta}\delta_{\alpha,\beta}\,r^*_\alpha\,a^\dagger_\beta a_\alpha\,, \tag{6}$$

so that the commutation relations are given by

$$[\tau_+,\tau_-] = \sum_\alpha |r_\alpha|^2\,a^\dagger_\alpha a_\alpha - \sum_\beta |r_\beta|^2\,a^\dagger_\beta a_\beta \neq \tau_0\,, \tag{7}$$

which shows explicitly that the Standard Model isospin commutation relations are violated if one uses the isospin operator of TH. The violation is due to neglecting parts of the proton wave functions that are in radial excitations when expanded in a neutron basis. This formal problem motivated us to develop an exact formalism for ISB corrections and to study corrections to the TH treatment [5].

For the calculation of δ_C, TH [2] proceed by introducing into Eq. (3) a complete set of states for the $(A-1)$-particle system, $|\pi\rangle$, which leads to

$$M_F = \sum_{\alpha,\pi}\langle f|a^\dagger_\alpha|\pi\rangle\langle\pi|a_\alpha|i\rangle\,r^\pi_\alpha\,. \tag{8}$$

The TH model thus allows for a dependence of the radial integrals on the intermediate state π. If isospin were an exact symmetry, the matrix elements of the creation and annihilation operators would be related by hermiticity, $\langle\pi|a_\alpha|i\rangle = \langle f|a^\dagger_\alpha|\pi\rangle^*$, and all radial integrals would be unity. Hence the symmetry-limit matrix element in the TH model is given by

$$M_0 = \sum_{\alpha,\pi}|\langle f|a^\dagger_\alpha|\pi\rangle|^2\,. \tag{9}$$

Towner and Hardy divide the contributions from ISB into two terms. First, the hermiticity of the matrix elements of a_α and $a^\dagger_\alpha$ will be broken, and second, the radial integrals will differ from unity. Assuming both effects are small, TH separate the resulting ISB corrections into two model-dependent parts [2]

$$\delta_C = \delta_{C1} + \delta_{C2}\,, \tag{10}$$

where in evaluating δ_{C1} all radial integrals are set to unity but the matrix elements are not assumed to be related by hermiticity, and in evaluating δ_{C2} it is assumed that $\langle\pi|a_\alpha|i\rangle = \langle f|a^\dagger_\alpha|\pi\rangle^*$ but $r^\pi_\alpha \neq 1$.

While Eq. (3) accounts for important effects of the Coulomb interaction on the radial wave functions, radial excitations are neglected in Eq. (4). We will show in Sect. IV that radial excitations contribute at the same (second) order in ISB interactions and are estimated to decrease the ISB corrections δ_{C2} obtained using the TH isospin operator.

III. EXACT FORMALISM AND THEOREMS FOR ISB CORRECTIONS

In this section, we recall aspects of the formalism and theorems of Ref. [5]. These show that there are no first-order ISB corrections to Fermi matrix elements and provide a basis for a complete second-order calculation. The formalism starts from the correct isospin operator

$$\tau_+ = \sum_\alpha a^\dagger_\alpha b_\alpha\,, \tag{11}$$

where α represents any single-particle basis, and $a^\dagger_\alpha$ creates neutrons and b_α annihilates protons in state α. The Fermi matrix element is then given by

$$M_F = \langle f|\tau_+|i\rangle\,, \tag{12}$$

where $|i\rangle$ and $|f\rangle$ are the exact initial and final eigenstates of the full Hamiltonian $H = H_0 + V_C$, with energy E_i and E_f, respectively. Here V_C denotes the sum of all interactions that do not commute with the vector isospin operator $\mathbf{T} = \sum_{i=1}^{A} \boldsymbol{\tau}_i/2$,

$$[H, \mathbf{T}] = [V_C, \mathbf{T}] \neq 0 \quad \text{and} \quad [H_0, \mathbf{T}] = 0\,. \tag{13}$$

We use round states, $|i)$ and $|f)$, to denote the bare initial and final eigenstates of H_0. Obtaining these states requires a solution of the A-body problem.

The full initial and final eigenstate $|i\rangle$ and $|f\rangle$ can then be written as

$$|i\rangle \;=\; \sqrt{Z_i}\left[|i) + \frac{1}{E_i - \Lambda_i H \Lambda_i}\,\Lambda_i V_C\,|i)\right], \tag{14}$$

$$|f\rangle \;=\; \sqrt{Z_f}\left[|f) + \frac{1}{E_f - \Lambda_f H \Lambda_f}\,\Lambda_f V_C\,|f)\right], \tag{15}$$

with projectors $\Lambda_i \equiv 1 - |i)(i|$ and $\Lambda_f = 1 - |f)(f|$. The unperturbed states $|i)$ and $|f)$ are related by an isospin rotation, so that $|f)$ is the isobaric analog state of $|i)$:

$$|f) = \frac{1}{\sqrt{2T}}\,\tau_+\,|i) \quad \text{and} \quad |i) = \frac{1}{\sqrt{2T}}\,\tau_-\,|f)\,, \tag{16}$$

where T denotes the isospin of the states $|i)$ and $|f)$, and for simplicity we consider parent cases with isospin projection $T_z = (N-Z)/2 = -T$ or $T = 1$, $T_z = 0$, so that $M_0 = (f|\tau_+|i) = \sqrt{2T}$. The factors Z_i and Z_f are taken to be real and ensure that the full eigenstates are normalized. With $(f|\,\tau_+\Lambda_i = 0$ and $\Lambda_f\tau_+\,|i) = 0$, the exact Fermi matrix element is given by

$$M_F = \sqrt{Z_i Z_f}\,\Bigg[M_0 + (f|\,V_C\Lambda_f\,\frac{1}{E_f - \Lambda_f H \Lambda_f} \times \tau_+\,\frac{1}{E_i - \Lambda_i H \Lambda_i}\,\Lambda_i V_C\,|i)\Bigg]. \tag{17}$$

This is the first theorem [5] and, because of $Z_{i,f} = 1 + \mathcal{O}(V_C^2)$, demonstrates that there are no first-order ISB corrections to M_F.

Another formulation is obtained by expanding in the difference of the charge-dependent interactions ΔV_C between the initial proton-rich and final neutron-rich states. In this case, the full Hamiltonian is given by

$$H = \widetilde{H}_0 + \Delta V_C\,, \tag{18}$$

where $\widetilde{H}_0$ includes the effects of V_C common to the initial and final states, for example the Coulomb interactions in the core, and ΔV_C takes into account all charge-dependent interactions of the extra proton with the other nucleons in the initial state. In this formulation, the bare states, $|i)$ and $|f)$, are eigenstates of $\widetilde{H}_0$, but are not eigenstates of the isospin operator:

$$|i) \;=\; \sum_{T' \geqslant |T_z|} \gamma_{T'}\,|T', T_z)\,, \tag{19}$$

$$|f) \;=\; \sum_{T' \geqslant |T_z+1|} \gamma_{T'}\,|T', T_z+1)\,. \tag{20}$$

In the isospin-symmetry limit, $\gamma_{T'} = \delta_{T',T}$, where T is the isospin of the bare states of Eq. (16) (which is also the dominant isospin in the presence of ISB).

In this case, the full eigenstates can be written as

$$|f\rangle = |f) \quad \text{and} \quad |i\rangle = \sqrt{Z}\,|i) + \frac{1}{E_i - \Lambda_i \widetilde{H}_0 \Lambda_i}\,\Lambda_i \Delta V_C\,|i\rangle\,, \tag{21}$$

and we obtain for the exact Fermi matrix element

$$M_F = \sqrt{Z}\,\sum_{T'} |\gamma_{T'}|^2 \sqrt{T'(T'+1) - T_z(T_z+1)} + (f|\tau_+\Lambda_i\,\frac{1}{E_i - \Lambda_i \widetilde{H}_0 \Lambda_i}\,\Lambda_i\,\Delta V_c\,|i\rangle\,, \tag{22}$$

where the sum is over $T' \geqslant \max(|T_z|, |T_z+1|)$. The $|T, T_z)$ expansion of the states $|i)$ and $|f)$ presents a more careful evaluation of the second theorem of Ref. [5]. Since $\gamma_T = 1 + \mathcal{O}(V_C^2)$ and $\gamma_{T' \neq T} = \mathcal{O}(V_C)$, it follows that $(f|\tau_+\Lambda_i$ is of first order in ISB interactions. Combined with $Z = 1 + \mathcal{O}\big((\Delta V_C)^2\big)$, Eq. (22) explicitly shows that ISB corrections to M_F start at second order. In the following, we will work with the first formulation.

IV. RELATION BETWEEN TH OPERATOR AND ISOSPIN

Next we derive a relation between the TH operator and the correct isospin operator based on the exact formalism. We use basis states given by conveniently-chosen one-body potentials U and $U + U_C$, where U_C accounts for charge-dependent effects. The single-particle (sp) potentials are chosen to minimize the effects of residual interactions,

$$\Delta V = V + V_C - (U + U_C)\,, \tag{23}$$

so that the Hamiltonian is given by

$$H = H_{\rm sp} + \Delta V \quad \text{with} \quad H_{\rm sp} = T + U + U_C\,. \tag{24}$$

We express the isospin raising operator τ_+ in a mixed representation, where $|\alpha\rangle$ denotes the eigenstates of the single-particle Hamiltonian $H_{\rm sp}$ and $|\widetilde{\alpha}\rangle$ the eigenstates of the isospin-symmetric part $T+U$. The creation operators in the two bases are related by

$$a^\dagger_\alpha = \sum_{\alpha'} a^\dagger_{\widetilde{\alpha}'}\,\langle\widetilde{\alpha}'|\alpha\rangle\,, \tag{25}$$

where the tilde indicates the basis and the sum is over all single-particle quantum numbers. The correct isospin operator, Eq. (11), can then be expressed as

$$\tau_+ = \sum_{\alpha,\alpha'} a^\dagger_{\widetilde{\alpha}'}\,\langle\widetilde{\alpha}'|\alpha\rangle\,b_\alpha\,, \tag{26}$$

which TH use as their starting point. However, Eq. (26) allows the states $|\alpha\rangle$ and $|\widetilde{\alpha}'\rangle$ to have different radial quantum numbers n and n', to be explicit

$$\alpha = nljm \quad \text{and} \quad \alpha' = n'ljm\,, \tag{27}$$

with orbital angular momentum l, total angular momentum $j = l \pm 1/2$, and magnetic quantum number m.

The TH operator $\tau_+^{\rm TH}$ of Eqs. (3) and (4) is obtained by keeping the terms with $\alpha = \alpha'$,

$$\tau_+^{\rm TH} = \sum_\alpha a^\dagger_{\widetilde{\alpha}}\, b_\alpha\, r_\alpha \,, \tag{28}$$

with $r_\alpha = \langle\widetilde{\alpha}|\alpha\rangle$ in the TH notation. Therefore, we define the correction operator,

$$\delta\tau_+ = \sum_{\alpha,\alpha'\neq\alpha} a^\dagger_{\widetilde{\alpha}'}\, b_\alpha\, \langle\widetilde{\alpha}'|\alpha\rangle \,. \tag{29}$$

Then the correct isospin operator and the exact Fermi matrix element are given by

$$\tau_+ \;=\; \tau_+^{\rm TH} + \delta\tau_+ \,, \tag{30}$$

$$M_F \;=\; \langle f|\tau_+^{\rm TH}|i\rangle + \langle f|\delta\tau_+|i\rangle \,. \tag{31}$$

We evaluate both terms in Eq. (31) to second order in ISB interactions. This will explicitly demonstrate that the second term due to radial excitations is of the same (second) order as the TH term. We start with the latter,

$$M_F^{\rm TH} = \langle f|\tau_+^{\rm TH}|i\rangle = M_0 - \langle f|\sum_\alpha a^\dagger_{\widetilde{\alpha}}\, b_\alpha\,(1-r_\alpha)|i\rangle \,. \tag{32}$$

Here we have applied the TH procedure and neglected the isospin-mixing correction δ_{C1} of Eq. (10), which is small in Ref. [2], so that $M_F^{\rm TH} = M_0$ for $r_\alpha = 1$. Since $(1-r_\alpha)$ starts at second order in ISB interactions, we can replace the full eigenstates in the second term in Eq. (32) by $|i)$ and $|f)$. Using a single-particle version of Eq. (21), we express $(1-r_\alpha)$ in terms of the matrix elements of the one-body potential U_C, that accounts for the difference between the $|\widetilde{\alpha}\rangle$ and $|\alpha\rangle$ basis states, to second order,

$$1 - r_\alpha \approx \frac{1}{2}\sum_{\alpha'\neq\alpha} \frac{\big|\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle\big|^2}{\big(\widetilde{E}_\alpha - \widetilde{E}_{\alpha'}\big)^2} \,, \tag{33}$$

where $\widetilde{E}_\alpha$ are the eigenvalues of the isospin-symmetric single-particle Hamiltonian $T+U$. For simplicity, we take $|i)$ to be a $Z-N$ proton plus core configuration. We define the occupation probabilities $\widetilde{\rho}_\alpha$ of the proton excess in the $|\widetilde{\alpha}\rangle$ basis, normalized so that $\sum_\alpha \widetilde{\rho}_\alpha = (i|\tau_-\tau_+|i) = 2T$.[1] As a result, we find for the TH term

$$M_F^{\rm TH} \approx M_0 - \frac{1}{2}\frac{1}{\sqrt{2T}}\sum_{\alpha,\alpha'\neq\alpha} \widetilde{\rho}_\alpha\, \frac{\big|\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle\big|^2}{\big(\widetilde{E}_\alpha - \widetilde{E}_{\alpha'}\big)^2} \,. \tag{36}$$

[1] To clarify the notation, the symmetry-limit matrix element can then be written as

$$M_0 \;=\; \frac{1}{\sqrt{2T}}\,(i|\sum_\alpha b^\dagger_{\widetilde{\alpha}}\, a_{\widetilde{\alpha}} \sum_\beta a^\dagger_{\widetilde{\beta}}\, b_{\widetilde{\beta}}\,|i) \,, \tag{34}$$

$$\;=\; \frac{1}{\sqrt{2T}}\,(i|\sum_\alpha b^\dagger_{\widetilde{\alpha}}\, b_{\widetilde{\alpha}}\,|i) = \frac{1}{\sqrt{2T}}\sum_\alpha \widetilde{\rho}_\alpha \,, \tag{35}$$

where the $|\widetilde{\alpha}\rangle$ basis, appropriate for the state $|i)$, was used for τ_+ and τ_-.

In the limit of sharp occupation probabilities $\widetilde{\rho}_\alpha$, the α (α') sum is over occupied (unoccupied) states.

Next we evaluate the contributions due to radial excitations, $\delta M_F = \langle f|\delta\tau_+|i\rangle$. To second order in ISB interactions, we have

$$\begin{aligned}\delta M_F \;\approx\;& (f|\delta\tau_+|i) + \frac{1}{\sqrt{2T}}\,(i|\,\tau_-\,\delta\tau_+\frac{1}{E_i - \Lambda_i H\Lambda_i}\,\Lambda_i V_C|i) \\ &+\; \frac{1}{\sqrt{2T}}\,(i|\,\tau_-\,V_C\Lambda_f\,\frac{1}{E_f - \Lambda_f H\Lambda_f}\,\delta\tau_+\,|i) \,.\end{aligned} \tag{37}$$

While we will estimate δM_F making similar approximations as for $M_F^{\rm TH}$ of Eq. (36), the result Eq. (37) provides a correction term that can be included in future numerical calculations of ISB corrections. For $\delta\tau_+$ we also need the overlap $\langle\widetilde{\alpha}'|\alpha\rangle$ for $\alpha'\neq\alpha$, which starts at first order,

$$\langle\widetilde{\alpha}'|\alpha\rangle \approx \frac{1}{\widetilde{E}_\alpha - \widetilde{E}_{\alpha'}}\,\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle \,. \tag{38}$$

We start with the first term of Eq. (37),

$$\begin{aligned}(f|\delta\tau_+|i) \;&=\; \frac{1}{\sqrt{2T}}\,(i|\sum_\beta b^\dagger_{\widetilde{\beta}}\, a_{\widetilde{\beta}} \sum_{\alpha,\alpha'\neq\alpha} a^\dagger_{\widetilde{\alpha}'}\, b_\alpha\, \langle\widetilde{\alpha}'|\alpha\rangle\,|i) \,, \\ &=\; \frac{1}{\sqrt{2T}}\,(i|\sum_{\alpha,\alpha'\neq\alpha} b^\dagger_{\widetilde{\alpha}'}\, b_\alpha\,\langle\widetilde{\alpha}'|\alpha\rangle\,|i) \,,\end{aligned} \tag{39}$$

where we have used that, for the considered configuration of $|i)$, the neutron annihilation and creation operators evaluate to $\delta_{\widetilde{\alpha},\widetilde{\beta}}$. After transforming b_α to the $|\widetilde{\alpha}\rangle$ basis, using the Hermitian conjugate of Eq. (25), we obtain

$$\begin{aligned}(f|\delta\tau_+|i) \;&=\; \frac{1}{\sqrt{2T}}\,(i|\sum_{\alpha,\alpha'\neq\alpha} b^\dagger_{\widetilde{\alpha}'} \sum_\beta b_{\widetilde{\beta}}\,\langle\alpha|\widetilde{\beta}\rangle\,\langle\widetilde{\alpha}'|\alpha\rangle\,|i) \,, \\ &\approx\; \frac{1}{\sqrt{2T}}\sum_{\alpha,\alpha'\neq\alpha} \widetilde{\rho}_{\alpha'}\,\frac{\big|\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle\big|^2}{\big(\widetilde{E}_\alpha - \widetilde{E}_{\alpha'}\big)^2} \,.\end{aligned} \tag{40}$$

We estimate the second and third terms of Eq. (37) using a closure approximation, that is we replace $E_i - \Lambda_i H\Lambda_i$ by $\Delta E_i < 0$, and similarly for $E_f - \Lambda_f H\Lambda_f$. In addition, we approximate V_C by the ISB one-body potential $\sum_{\gamma,\gamma'\neq\gamma}\langle\gamma|U_C|\gamma'\rangle\, b^\dagger_\gamma\, b_{\gamma'}$, where $\gamma\neq\gamma'$ ensures the action of the projectors Λ_i and Λ_f. After contracting the neutron operators, we find for the second term

$$\begin{aligned}&\frac{1}{\sqrt{2T}}\,(i|\,\tau_-\,\delta\tau_+\frac{1}{E_i - \Lambda_i H\Lambda_i}\,\Lambda_i V_C\,|i) \\ &=\; \frac{1}{\sqrt{2T}}\,(i|\sum_{\alpha,\alpha'\neq\alpha} b^\dagger_{\widetilde{\alpha}'} b_\alpha\,\frac{\langle\widetilde{\alpha}'|\alpha\rangle}{\Delta E_i}\sum_{\gamma,\gamma'\neq\gamma}\langle\gamma|U_C|\gamma'\rangle\, b^\dagger_\gamma b_{\gamma'}|i) \,, \\ &=\; -\frac{1}{\sqrt{2T}}\sum_{\alpha,\alpha'\neq\alpha} \widetilde{\rho}_{\alpha'}\,\frac{\big|\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle\big|^2}{\big|(\widetilde{E}_\alpha - \widetilde{E}_{\alpha'})\Delta E_i\big|} \,.\end{aligned} \tag{41}$$

For the third term, we obtain

$$\frac{1}{\sqrt{2T}}\,(i|\,\tau_- \, V_C \Lambda_f \,\frac{1}{E_f - \Lambda_f H \Lambda_f}\,\delta\tau_+\,|i)$$
$$= \frac{1}{\sqrt{2T}}\,(i|\sum_{\alpha,\alpha'\neq\alpha} b^\dagger_{\widetilde{\alpha}'} \sum_{\gamma,\gamma'\neq\gamma} \langle\gamma|U_C|\gamma'\rangle\, b^\dagger_\gamma\, b_{\gamma'} b_\alpha\,\frac{\langle\widetilde{\alpha}'|\alpha\rangle}{\Delta E_f}|i)\,.$$

For nonzero overlap, this requires two-particle–two-hole configurations in $|i)$, while the first and second terms receive contributions at the level of the best Slater determinant. Assuming residual interactions are weak, we neglect the third term. Combining Eqs. (40) and (41), we have for the correction term

$$\delta M_F = \frac{1}{\sqrt{2T}} \sum_{\alpha,\alpha'\neq\alpha} \widetilde{\rho}_{\alpha'}\,\frac{\big|\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle\big|^2}{\big(\widetilde{E}_\alpha - \widetilde{E}_{\alpha'}\big)^2} - \frac{1}{\sqrt{2T}} \sum_{\alpha,\alpha'\neq\alpha} \widetilde{\rho}_{\alpha'}\,\frac{\big|\langle\widetilde{\alpha}'|U_C|\widetilde{\alpha}\rangle\big|^2}{\big|(\widetilde{E}_\alpha - \widetilde{E}_{\alpha'})\Delta E_i\big|}\,. \quad (42)$$

Comparing our estimate δM_F to the corresponding TH term, Eq. (36), demonstrates that radial excitations are significant. The same estimate, Eq. (42), is found when V_C is approximated by the isovector part of the ISB one-body potential, $\sum_{\gamma,\gamma'\neq\gamma}\langle\gamma|U_C|\gamma'\rangle\,(b^\dagger_\gamma\, b_{\gamma'} - a^\dagger_\gamma\, a_{\gamma'})/2$. In this case, the second term of δM_F is 1/2 of Eq. (41), but the third term also yields 1/2 of this (for $\Delta E_i = \Delta E_f$).

Assuming radial excitations are dominated by n to $n+1$, we have $\widetilde{E}_\alpha - \widetilde{E}_{\alpha'} = 2\hbar\omega$, where ω is a typical oscillator frequency. Moreover, if the relevant excitations are dominated by the isovector monopole state, the value of $|\Delta E_i|$ ranges between 3 and $4\hbar\omega$ [8]. For $\widetilde{E}_\alpha - \widetilde{E}_{\alpha'} = 2\hbar\omega$ and $|\Delta E_i| = 4\hbar\omega$, we find that the correction term completely cancels the TH contribution δ_{C2} at second order. This result is similar to the energy-degenerate case of Auerbach [8]. Our estimate shows that, if the contributions of the isovector monopole state dominate ISB, the radial excitations neglected by TH decrease ISB corrections.

V. IMPLICATIONS FOR ISB CORRECTIONS

We have used the exact formalism of Ref. [5] to calculate the renormalization of the Fermi matrix element due to radial contributions. We expressed the correct isospin operator as a sum of the TH operator and a correction term involving radial excitations, which were shown to be significant and estimated to decrease the ISB corrections δ_{C2} of Ref. [2]. In addition, for certain conditions the ISB corrections due to radial contributions can cancel at second order in ISB interactions. A reduction due to the correction term implies that the extracted value of V_{ud} may be reduced. Moreover, our results can provide a possible explanation for the smaller ISB corrections found in Refs. [8, 9], although these calculations are more exploratory at this stage.

An important direction for future research is to include the correction term of Eqs. (29) and (37) in numerical calculations of ISB corrections that follow the TH approach or make a similar truncation of basis states.

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